

ON THE STIELTJES INTEGRAL

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INTRODUCTION

0.01 The ordinary definition of the Stieltjes integral of a function $f(x)$ with respect to $g(x)$ on the interval from a to b , $a < b$, is as follows:- let

$$\sigma \equiv \{a = x_0 < x_1 < \dots < x_i < \dots < x_m = b\}$$

be a subdivision of (a, b) , and consider the sum

$$\theta(\sigma) \equiv \sum_{i=1}^m f(\xi_i) [g(x_i) - g(x_{i-1})], \quad x_{i-1} \leq \xi_i \leq x_i.$$

$\theta(\sigma)$ thus has in general many values for a given σ , depending upon the "inner" points ξ_i . Suppose now that the sums $\theta(\sigma)$, which may be called "Stieltjes" sums, approach a finite number A as limit when the norm of σ approaches 0 as limit; where by the norm of a σ is meant the maximum distance between two consecutive subdivision points of it. In such a case one says that the integral of $f(x)$ with respect to $g(x)$ on (a, b) is A , and denotes this by writing

$$\int_a^b f(x) dg(x) = A.$$

0.02 From one point of view, the two essential features of the above process are:

1) the definition, by means of two functions already defined on an interval, of a many-valued numerical function of certain (finite) subsets σ of that interval; - i.e., the Stieltjes sums above; - and

2) The application of a certain limiting process (above, the limit according to the norm of σ) to this new function.

Hence, one may generalize the process of Stieltjes, and the consequent definition, by starting at once with a set-function as already given; that is, one may suppose that with every set σ of a family of sets S there is already associated one or more numbers, the values of the function for the given set σ . Further, instead of assigning a "norm" to every σ , one may take some other relation between the elements of S as the basis of a limiting process.

This point of view is not new. Miss Young used many-valued sums (such as $\theta(\sigma)$ above) in a series of papers.¹ H. L. Smith,² S. Pollard,³ and A. Kolmogoroff⁴ considered integrals where the passage to the limit was in the σ -sense (as explained below, 0.03).

0.03 The present paper starts with a study of the consequences of generalizing, in accordance with the last paragraph, the process (and definition) of Stieltjes as given in 0.01.

More precisely, in Chapter I there is first described a class S of (finite) point-sets σ taken from a space X of points which are arranged in a linear order. Some pairs of elements of S are subject to the relation "finer than." The numerical many-valued set-functions $\theta(\sigma)$ which will be considered are then defined as possessing what may be called the "additivity" property. In terms of these notions, the analogue of the Stieltjes integral is then defined as follows: the number A will be the σ -limit of the additive set-function θ on (c,d) [where c and d are points of X], if for every positive ϵ there exists a σ_ϵ , with end-points c and d , such that

$$|\theta(\sigma) - A| \leq \epsilon$$

for all the σ 's which are finer than σ_ϵ , and for all possible determinations of $\theta(\sigma)$.

0.04 After some of the simpler properties of this σ -limit are stated, one comes to a theorem which is closely analogous to the theorem on integration by parts in the classical theory. It is shown that if three set-functions are "related by parts", then the existence of the σ -limit of any two of these functions implies the existence of the σ -limit of the third one. Furthermore, it appears from the proof that the theorem on integration by parts is in effect an extension of the rather simple theorem that the integral of the sum of two functions is defined if the integral of each of the summands exists.

0.05 Almost all the theorems in the theory of Stieltjes integration which state under what conditions the limit function of a set of integrable functions is integrable require, in one form or another, uniformity of convergence. In connection with similar theorems regarding the σ -limit, it is shown toward the end of Chapter I that this uniformity may be required either of the set-functions themselves, or else of certain other (set) functions associated with the former in a natural way.

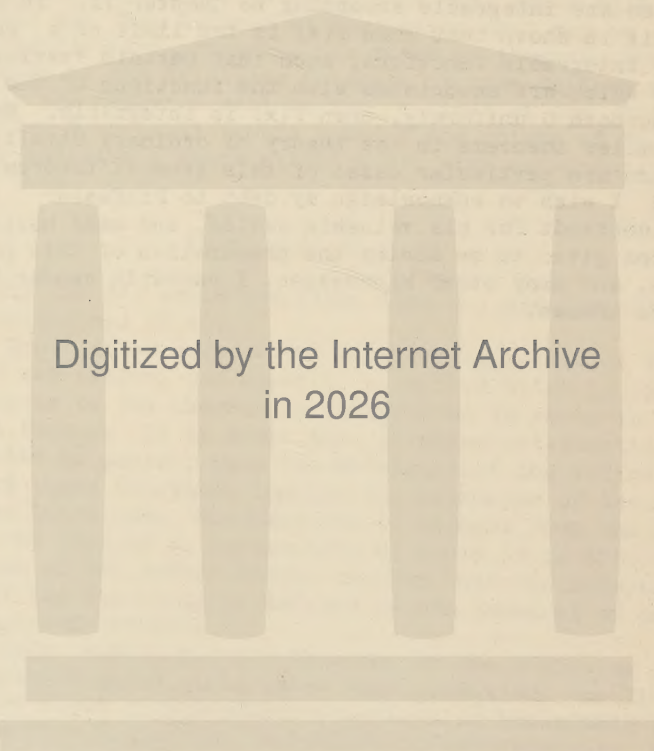
0.06 Chapter II interprets the results obtained therefore in terms of a number of specific examples.

Thus, there is first discussed a (Stieltjes) integral of a function $f(x)$ with respect to a function $g(x)$ on a linear interval (a,b) , except that the σ -wise passage to the limit is used instead of the "norm" limit described in 0.01. Another sort of an integral is obtained when, in addition to

changing the limiting process, one also changes the definition of the sums $\theta(\sigma)$ [so that the "inner" points ξ_i are required to be interior to the interval (x_{i-1}, x_i)].⁵ Finally, by applying the new limiting process to an integral of a (single) function $K(x,y)$ considered incidentally by Fischer,⁶ one obtains still another interpretation of the results of Chapter I.

0.07 Finally, Chapter III considers sequences of functions which are integrable according to Chapter II. In particular, it is shown that when $f(x)$ is the limit of a sequence of integrable functions, such that certain "variation" functions which are associated with the functions of the sequence approach 0 uniformly, then $f(x)$ is integrable. Many of the similar theorems in the theory of ordinary Stieltjes integration are particular cases of this general theorem.

0.08 I wish to acknowledge my debt to Professor T. H. Hildebrandt for his valuable advice, and many helpful suggestions given to me during the preparation of this paper. For these, and many other kindnesses, I herewith render him my sincere thanks.



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CHAPTER I

A. DEFINITION OF THE σ -LIMIT. CONDITIONS FOR ITS EXISTENCE

1.01 We will consider a space X of elements $x, y, z, \dots$; these elements are ordered in such a way that, of any two different elements, x and y , one "precedes" the other, - in symbols $x < y$ (or $y < x$). This ordering relation is such that, if $x < y$, then y does not precede x ; finally, if $x < y$ and $y < z$, then $x < z$.

1.02 Let S denote the family of all finite subsets, σ or s , of X which contain two or more elements. The $k + 1$ elements of a σ will be denoted by

$$x_0, x_1, \dots, x_k,$$

the notation being such that $x_i < x_{i+1}$ ($i = 0, \dots, k-1$). x_0 and x_k will be referred to as the initial and terminal elements of σ , respectively; together, they will be called the end-elements of σ .

1.03 The notation $\sigma \geq \sigma'$ to be read " σ is finer than σ' " will denote that σ and σ' have the same end-elements and that every element of σ' is also an element of σ . If σ and σ' have the same end-elements, then $\sigma \cdot \sigma'$ will denote the set of all the elements in σ and σ' . If the terminal element of σ is the same as the initial element of σ' , then $\sigma + \sigma'$ will be used to denote the set composed of all the elements in σ and σ' . The symbol $\sum_1^n \sigma^i$ is defined in a similar fashion, for any finite n . If

$$\sigma = \{x_0 < x_1 < \dots < x_{i-1} < x_i < \dots < x_k\}$$

and

$$s^i = \{x_{i0} < \dots < x_{im}\},$$

and $x_{i0} = x_{i-1}$, $x_{im} = x_i$, then s^i will be referred to as a subdivision of the i th mesh of σ . If there is given such a subdivision for each mesh of σ , say $s^1, s^2, \dots, s^k$, then clearly

$$\sum_1^k s^i \geq \sigma.$$

It should be noted that there is at least one subdivision for any mesh of σ .

The following facts may be stated without proof for the sake of completeness:

- a) The relation $\geq$ on S is reflexive and transitive.
- b) $\sigma \cdot \sigma' \geq \sigma$; $\sigma \cdot \sigma' \geq \sigma'$.
- c) If $\sigma'' \geq \sigma \cdot \sigma'$, then $\sigma'' \geq \sigma$ and $\sigma'' \geq \sigma'$.

1.04 Let θ be a many-valued⁷ number - valued function on S (the symbols ∞ and $-\infty$ being admitted as possible values, with the usual interpretations and conventions on the meaning of $\infty + \infty$, etc.). We will denote a general value of θ for a given σ by $\theta(\sigma)$; if it should become necessary to refer to some particular one of these values, we will denote it by $\theta^0(\sigma)$ or $\theta^\alpha(\sigma)$, etc. We require further that the function θ have the "additive" property that if $\sigma = s^1 + s^2$, then

$$\theta(\sigma) = \theta(s^1) + \theta(s^2);$$

i.e., for a given value of $\theta(\sigma)$, there are values of $\theta(s^1)$ and $\theta(s^2)$ --- and conversely, for given values of $\theta(s^1)$ and $\theta(s^2)$, there is a value of $\theta(\sigma)$ --- such that the indicated equality holds. Finally, let it be agreed that whenever we shall use the word set-function in what follows, it will mean a function of the sort just described, i.e., an "additive" set-function.

1.05 Considering only those sets σ whose initial and terminal elements are c and d , respectively, it may happen that there exists a finite number A with the property that for every positive ϵ , however small, we can find a σ_ϵ with end-elements c and d such that

$$|\theta(\sigma) - A| \leq \epsilon,$$

for all $\sigma \geq \sigma_\epsilon$ and all possible values of $\theta(\sigma)$. In such a case we will say that the σ -limit of θ on (c, d) is A , and we will symbolize this by writing

$$\lim_{\sigma} \theta(c, d) = A.$$

If there is no danger of confusion, the phrase "on (c, d) " will be omitted.

We will now state a number of theorems dealing with the σ -limit. For the most part, proofs will be omitted, these being either fairly obvious or else following closely the proofs given in the indicated references; where the proof in the reference is given for the case of the "norm" - limit,

the required modification will be found to be quite easy.

1.06 Theorem:⁸ If θ has a σ -limit, then it has only one such limit.

1.07 Theorem:⁹ A necessary and sufficient condition for the existence of $\lim_{\sigma} \theta(c,d)$ is that for every $\epsilon > 0$, we can find a c^{ϵ} with end-elements c and d such that

$$|\theta(\sigma') - \theta(\sigma'')| \leq \epsilon,$$

for all σ' and σ'' finer than c^{ϵ} .

1.08 If g and h are elements of X , $g < h$, consider the LUB¹⁰ (which may be ω) of the numbers $|\theta(\sigma') - \theta(\sigma'')|$, for all σ', σ'' whose end-elements are g and h . We will denote this LUB by $O(\theta, g, h)$ and call it the oscillation of θ on (g, h) . If σ is any set in S :

$$\sigma \equiv \{x_0 < x_1 < \dots < x_k\}$$

we define

$$V(\theta, \sigma) = \sum_1^k O(\theta, x_{i-1}, x_i)$$

and call it the variation of θ on σ . This function on S (for a given θ) is obviously a function of the same sort as θ itself. Using this "variation" - function, we can state some additional theorems about the σ -limit and the function θ .

1.091 Lemma: If $\sigma', \sigma'' \geq \sigma$, then

$$|\theta(\sigma') - \theta(\sigma'')| \leq V(\theta, \sigma).$$

1.09 Theorem: If $\lim_{\sigma} \theta(c,d)$ exists, then for any σ (with end-elements c and d),

$$|\lim_{\sigma} \theta(c,d) - \theta(\sigma)| \leq V(\theta, \sigma).$$

1.101 Lemma: For every σ and $h > 0$, we can find $\sigma', \sigma'' \geq \sigma$ such that

$$|\theta(\sigma') - \theta(\sigma'')| > V(\theta, \sigma) - h.$$

1.10 Theorem:¹¹ A necessary and sufficient condition for the existence of $\lim_{\sigma} \theta(c,d)$ is that

$$\lim_{\sigma} V(\theta, c, d) = 0.$$

B. CONSEQUENCES OF THE DEFINITION
OF THE σ -LIMIT.

2.01 Theorem: If $c < d < f$, then.

a) If $\lim_{\sigma} \theta(c, f)$ exists, then $\lim_{\sigma} \theta(c, d)$ and $\lim_{\sigma} \theta(d, f)$ will both exist;

b) If $\lim_{\sigma} \theta(c, d)$ and $\lim_{\sigma} \theta(d, f)$ both exist, then $\lim_{\sigma} \theta(c, f)$ will also exist;

--- in both cases we will have, moreover,

$$\lim_{\sigma} \theta(c, d) + \lim_{\sigma} \theta(d, f) = \lim_{\sigma} \theta(c, f).$$

Since statement b) does not hold for the "norm"-limit, we will give a

Proof for b): Put $\lim_{\sigma} \theta(c, d) = A$ and $\lim_{\sigma} \theta(d, f) = B$. Let $\epsilon > 0$ be chosen, and let $\bar{\sigma}^e$, with end-elements c, d , and $\bar{\sigma}^e$, with end-elements d, f , be such that

$$(I) \begin{cases} |\theta(\sigma) - A| \leq \frac{\epsilon}{2}; & \sigma \geq \bar{\sigma}^e; \\ |\theta(\sigma) - B| \leq \frac{\epsilon}{2}; & \sigma \geq \bar{\sigma}^e. \end{cases}$$

If we define $\sigma^e = \bar{\sigma}^e + \bar{\sigma}^e$, and consider any $\sigma \geq \sigma^e$, then clearly σ may be looked upon as the sum of two sets σ' and σ'' --

$$\sigma = \sigma' + \sigma'', \text{ ---}$$

where $\sigma' \geq \bar{\sigma}^e$ and $\sigma'' \geq \bar{\sigma}^e$. For any such σ we will thus have:

$$\begin{aligned} |\theta^o(\sigma) - (A + B)| &= |\theta^o(\sigma') - A + \theta^o(\sigma'') - B| \leq \\ &\leq |\theta^o(\sigma') - A| + |\theta^o(\sigma'') - B| \leq \epsilon, \end{aligned}$$

by (I), which means that $\lim_{\sigma} \theta(c, f)$ exists and equals $A + B$.

2.02 If we define

$$\lim_{\sigma} \theta(d, e) = - \lim_{\sigma} \theta(e, d); \quad e < d;$$

and

$$\lim_{\sigma} \theta(a, a) = 0,$$

then we may state, because of theorem 2.01, the following

Corollary: For any three elements c, d, f in X ,

$$\lim_{\sigma} \theta(c, d) + \lim_{\sigma} \theta(d, f) + \lim_{\sigma} \theta(f, c) = 0,$$

whenever any two of the limits in question exist.

2.03 Having given two set-functions, θ_1 and θ_2 , we define θ_3 to be the function which is such that for every s and a value $\theta_3^0(s)$ there is a value $\theta_1^0(s)$ and $\theta_2^0(s)$ for which

$$\theta_3^0(s) = \theta_1^0(s) + \theta_2^0(s)$$

(though not necessarily conversely); in such a case we will write

$$\theta_3 = \theta_1 + \theta_2.$$

2.04 Theorem: If $\lim_{\sigma} \theta_1(c,d) = A$ and $\lim_{\sigma} \theta_2(c,d) = B$, and if $\theta_3 = \theta_1 + \theta_2$, then

$$\lim_{\sigma} \theta_3(c,d) = A + B.$$

2.05 We will write $\theta_1 = k\theta$ if the values which θ_1 assumes for a given s are those and only those which are k times the values of θ for the same s ; where k is any real number and θ is of the sort considered in 1.04.

2.06 Theorem: If $k \neq 0$, and if $\theta_1 = k\theta$, then $\lim_{\sigma} \theta_1(c,d) = k \lim_{\sigma} \theta(c,d)$, provided either one of the limits exists.

If $k = 0$, the theorem is still true if $\lim_{\sigma} \theta(c,d)$ exists.

2.07 Let three set-functions θ_1 , θ_2 , and θ_3 be such that for a given σ and a value of any one of them for this σ , say $\theta_1^0(\sigma)$, we can find $\theta_2^0(\bar{\sigma})$ and $\theta_3^0(\bar{\bar{\sigma}})$, where σ , $\bar{\sigma}$, $\bar{\bar{\sigma}}$ have the same end-elements, such that

$$\theta_1^0(\sigma) + \theta_2^0(\bar{\sigma}) + \theta_3^0(\bar{\bar{\sigma}}) = 0.$$

In such a case we shall write

$$\theta_1 + \theta_2 + \theta_3 = 0.$$

We may now state the following

Lemma: If

$$\theta_1 + \theta_2 + \theta_3 = 0,$$

and if the σ -limits of θ_1 and θ_2 on (c,d) both exist, then $\lim_{\sigma} \theta_3(c,d)$ also exists. Moreover,

$$\lim_{\sigma} \theta_1(c,d) + \lim_{\sigma} \theta_2(c,d) + \lim_{\sigma} \theta_3(c,d) = 0.$$

Proof: Because of 1.10, to prove the first part of our lemma it suffices to show that $\lim_{\sigma} V(\theta_3, c, d) = 0$, having given that $\lim_{\sigma} V(\theta_1, c, d)$ and $\lim_{\sigma} V(\theta_2, c, d)$ both vanish.

Let, then, $\epsilon > 0$ be given, and let σ^ϵ be such that

$$V(\theta_1, \sigma) \leq \epsilon, \quad V(\theta_2, \sigma) \leq \epsilon$$

for all $\sigma \geq \sigma^\epsilon$. Let $\sigma^0 \geq \sigma^\epsilon$, where

$$\sigma^0 = \{c = x_0 < x_1 < \dots < x_m = d\},$$

and let $\{s^i\}$ and $\{\sigma^i\}$, be any two subdivisions for the i th mesh of σ^0 . Then, by hypothesis,

$$\theta_3(\sigma^i) = -\theta_1(\bar{\sigma}^i) - \theta_2(\bar{\sigma}^i),$$

$$\theta_3(s^i) = -\theta_1(\bar{s}^i) - \theta_2(\bar{s}^i),$$

and so

$$\begin{aligned} |\theta_3(s^i) - \theta_3(\sigma^i)| &\leq |\theta_1(\bar{\sigma}^i) - \theta_1(\bar{s}^i)| + |\theta_2(\bar{\sigma}^i) - \theta_2(\bar{s}^i)| \leq \\ &\leq O(\theta_1, x_{i-1}, x_i) + O(\theta_2, x_{i-1}, x_i). \end{aligned}$$

whence

$$O(\theta_3, x_{i-1}, x_i) \leq O(\theta_1, x_{i-1}, x_i) + O(\theta_2, x_{i-1}, x_i)$$

and so, finally,

$$V(\theta_3, \sigma^0) \leq V(\theta_1, \sigma^0) + V(\theta_2, \sigma^0) \leq 2\epsilon$$

which proves that

$$\lim_{\sigma} V(\theta_3, c, d) = 0.$$

To prove the second part of the lemma, suppose now that $\lim_{\sigma} \theta_1(c, d) = A$, $\lim_{\sigma} \theta_2(c, d) = B$, $\lim_{\sigma} \theta_3(c, d) = C$. Let $\epsilon > 0$ be arbitrary, and let σ^ϵ be such that

$$V(\theta_i, \sigma^0) \geq \epsilon, \quad i = 1, 2, 3,$$

for any $\sigma^0 \leq \sigma^\epsilon$. Consider a particular σ^0 , say

$$\sigma^0 = \{c = x_0 < \dots < x_m = d\}$$

and put $\sigma^1 = \{x_{i-1}, x_i\}$. Then

$$\begin{aligned} \theta_1^\alpha(\sigma^0) &= \sum_{i=1}^m \theta_1^\alpha(\sigma^1) = - \left[\sum_{i=1}^m \theta_2^\beta(\bar{\sigma}^1) + \sum_{i=1}^m \theta_3^\gamma(\bar{\sigma}^1) \right] \\ &= - [\theta_2^\beta(\bar{\sigma}^0) + \theta_3^\gamma(\bar{\sigma}^0)], \end{aligned}$$

where $\bar{\sigma}^0 = \sum_{i=1}^m \bar{\sigma}^i$, $\bar{\sigma}^0 = \sum_{i=1}^m \bar{\sigma}^i$. Hence

$$\begin{aligned} |A + B + C| &\leq |A - \theta_1^\alpha(\bar{\sigma}^0)| + |B - \theta_2^\beta(\bar{\sigma}^0)| + |C - \theta_3^\gamma(\bar{\sigma}^0)| \leq \\ &\leq V(\theta_1, \bar{\sigma}^0) + V(\theta_2, \bar{\sigma}^0) + V(\theta_3, \bar{\sigma}^0) \leq 3e, \end{aligned}$$

by 1.09. Since $e > 0$ was arbitrary, we have

$$A + B + C = 0, \text{ i.e.,}$$

$$\lim_{\sigma} \theta_1(c, d) + \lim_{\sigma} \theta_2(c, d) + \lim_{\sigma} \theta_3(c, d) = 0.$$

2.08 Two set-functions θ_1 and θ_2 are said to be "related by parts", if for every $\theta_1^\alpha(s)$ there exists a $\theta_2^\beta(s')$ - and conversely, for every $\theta_2^\beta(s')$ there exists a $\theta_1^\alpha(s)$, - such that s and s' have the same end-elements, c and d , and

$$\theta_1^\alpha(s) + \theta_2^\beta(s') = k(c, d),$$

where k is a variable depending only on c and d , and, for $c < e < d$,

$$k(c, e) + k(e, d) = k(c, d).$$

2.09 Theorem: If θ_1 and θ_2 are related by parts, and if either of the σ -limits $\lim_{\sigma} \theta_1(c, d)$, $\lim_{\sigma} \theta_2(c, d)$ exists, then the other σ -limit also exists, and

$$\lim_{\sigma} \theta_1(c, d) + \lim_{\sigma} \theta_2(c, d) = k(c, d).$$

Proof: For any σ such that

$$\sigma \equiv \{\alpha = x_0 < x_1 < \dots < x_m = \beta\}$$

we define

$$\theta_3(\sigma) = -k(\alpha, \beta).$$

Then it is clear that θ_3 is an additive set-function such that

$$\lim_{\sigma} \theta_3(c, d) = -k(c, d),$$

and such that $\theta_1 + \theta_2 + \theta_3 = 0$. Hence, our theorem is a direct consequence of the lemma in 2.07.

2.10 If θ is a set-function, we define Φ as follows:
Consider

$$\sigma \equiv \{x_0 < \dots < x_m\}$$

and put

$$\sigma^1 = \{x_{i-1} < x_i\}; i = 1, \dots, m$$

Then

$$\Phi^o(\sigma) = \sum_1^m |\theta^1(\sigma^1)|,$$

where $\theta^1(\sigma^1)$ is, as explained already, one of the (possibly) many values of θ for σ^1 . Thus Φ is itself a set-function on S to the real numbers; all the values of Φ for σ are obtained by varying independently the values of θ for the several σ^1 .

2.11 Theorem:12 If for all α, β , such that

$$c \leq \alpha < \beta \leq d$$

we have

$$\lim_{\sigma} \theta(\alpha, \beta) = 0,$$

then for the same α, β

$$\lim_{\sigma} \Phi(\alpha, \beta) = 0.$$

[This theorem is known as Hyslop's Theorem, and is useful in considering a change of the variable in the σ -Stieltjes integral.]

2.12 Let θ_1 and θ_2 be two set-functions, and suppose that there is a σ_0 such that for every $\sigma < S$ for which $\sigma \geq \sigma_0$ the set of numbers $\theta_1(\sigma)$ is a subset of the set of numbers $\theta_2(\sigma)$. In such a case we shall say that θ_1 is "majorated" by θ_2 , and we shall denote this by writing $\theta_1 < \theta_2$.

2.13 Theorem: If $\theta_1 < \theta_2$, and if $\lim_{\sigma} \theta_2(c, d) = A$, then $\lim_{\sigma} \theta_1(c, d) = A$.

This is a direct consequence of the definition of the σ -limit.

C. CONCERNING SEQUENCES OF θ FUNCTIONS

3.01 Let θ be a set-function and let $\{\theta_n\}$ be a sequence of such functions. We shall say that the sequence $\{\theta_n\}$ converges to θ , if for every value $\theta^o(\sigma)$ there exist for each n a value $\theta_n^n(\sigma)$ such that

$$\lim_n \theta_n^n(\sigma) = \theta^o(\sigma),$$

this being true for every σ in S .

The variation-functions of a sequence of functions $\{\theta_n\}$ will be said to approach 0 uniformly, if for every $\epsilon > 0$ we can find a σ^ϵ (with the appropriate end-points) such that

$$V(\theta_n, \sigma) \leq \epsilon$$

for all $\sigma \geq \sigma^\epsilon$ and for all n .

3.02 Theorem: If on (c, d) the variation-functions of a sequence of functions $\{\theta_n\}$ approach 0 uniformly, and if the sequence $\{\theta_n\}$ converges to a function θ , then

$$\lim_{\sigma} V(\theta, c, d) = 0.$$

Proof: Let

$$\epsilon_1 > \epsilon_2 > \dots > \epsilon_k > \dots \rightarrow 0$$

and let σ^k , with end-elements c and d be such that for all $\sigma \geq \sigma^k$,

$$V(\theta_n, \sigma) \leq \epsilon_k \text{ for all } n; k = 1, 2, \dots$$

Let $\epsilon > 0$ be given; suppose

$$\epsilon_{j-1} \geq \epsilon > \epsilon_j$$

and consider any $\sigma \geq \sigma^j$, say

$$\sigma = \{c = x_0 < \dots < x_p = d\}.$$

Let s_i and t_i be any two subdivisions of the i th mesh of σ , and let $\theta'(s_i)$ and $\theta'(t_i)$ be any two values of θ for s_i and t_i . Because of the convergence, we can find a positive integer N_ϵ (which will depend, in general, on p and the s_i and t_i) such that for $n \geq N_\epsilon$ we will have

$$-\frac{\epsilon}{2p} + \theta_n^n(s_i) \leq \theta'(s_i) \leq \theta_n^n(s_i) + \frac{\epsilon}{2p},$$

and

$$-\frac{\epsilon}{2p} + \theta_n^n(t_i) \leq \theta'(t_i) \leq \theta_n^n(t_i) + \frac{\epsilon}{2p}$$

so that

$$|\theta'(s_i) - \theta'(t_i)| \leq |\theta_n^n(s_i) - \theta_n^n(t_i)| + \frac{\epsilon}{p} \leq 0(\theta_n, x_{i-1}, x_i) + \frac{\epsilon}{p}$$

Hence

$$\sum_1^p |\theta'(s_i) - \theta'(t_i)| \leq V(\theta_n, \sigma) + \epsilon; n \geq N_\epsilon$$

and since $\sigma \geq \sigma j$ (and so $V(\theta_n, \sigma) \leq \epsilon j < e$),

$$\sum_1^p |\theta'(s_1) - \theta'(t_1)| \leq 2e.$$

Since the several s_1 and t_1 , and the values of θ for these, were arbitrary, we may suppose that they are such that

$$|\theta'(s_1) - \theta'(t_1)| > 0(\theta, x_{i-1}, x_i) - \frac{e}{p}.$$

We will thus obtain finally

$$2e \geq \sum_1^p |\theta'(s_1) - \theta'(t_1)| > V(\theta, \sigma) - e,$$

or

$$3e \geq V(\theta, \sigma) \geq 0 \text{ for all } \sigma \geq \sigma j, \text{ --}$$

hence the theorem.

3.03 Theorem: Under the hypotheses of 3.02

$$\lim_{\sigma} \theta(c, d) = \lim_n \lim_{\sigma} \theta_n(c, d),$$

Proof: From the previous theorem and 1.10, we have at once that $\lim_{\sigma} \theta(c, d)$ exists, say $\lim_{\sigma} \theta(c, d) = A$, and that the other σ -limits also exist, say $\lim_{\sigma} \theta_n(c, d) = A_n$. Let σ be such that

$$V(\theta_n, \sigma) \leq \frac{e}{3} \text{ for all } n,$$

$$V(\theta, \sigma) \leq \frac{e}{3},$$

and let the positive integer N_e be such that for all $n \geq N_e$

$$|\theta_n^{\sigma}(\sigma) - \theta^{\sigma}(\sigma)| \leq \frac{e}{3}.$$

We may now write, for $n \geq N_e$,

$$\begin{aligned} |A_n - A| &\leq |A_n - \theta_n^{\sigma}(\sigma)| + |\theta_n^{\sigma}(\sigma) - \theta^{\sigma}(\sigma)| + |\theta^{\sigma}(\sigma) - A| \leq \\ &\leq V(\theta_n, \sigma) + \frac{e}{3} + V(\theta, \sigma) \leq e, \end{aligned}$$

whence $\lim_n A_n = A$.

3.04 We will say that a sequence of set-functions $\{\theta_n\}$ converges uniformly to a set-function θ on (c, d) , if for every $e > 0$ there exists a positive integer N_e such that both of the following conditions hold for all σ in S :

a) If $\theta^{\sigma}(\sigma)$ is a value of θ for σ and $n \geq N_e$, then we can find $\theta_n^{\sigma}(\sigma)$ such that

$$|\theta^0(\sigma) - \theta_n^n(\sigma)| \leq e;$$

b) If $n \geq N_e$ and $\theta_n^n(\sigma)$ is given, then we can find a value $\theta^0(\sigma)$ such that

$$|\theta_n^n(\sigma) - \theta^0(\sigma)| \leq e.$$

It may be noticed at once that uniform convergence entails ordinary convergence as defined in 3.01.

3.05 Theorem: If on (c, d) the sequence of set-functions $\{\theta_n\}$ converges uniformly to a set-function θ , and if $\lim_{\sigma} V(\theta_n, c, d) = 0$ for every n , then

$$\lim_{\sigma} V(\theta_n, c, d) = 0 \text{ uniformly.}$$

Proof: Let $e > 0$ be chosen. By the first part of our hypothesis, we can find a positive integer N such that a) and b) of 3.04 hold; by the second part of the hypothesis, we can find σ^n , with end-elements c and d , such that

$$V(\theta_n, \sigma) \leq e; \quad \sigma \geq \sigma^n; \quad n = 1, \dots, N, \dots$$

We define $\sigma^e = \sigma^1 \cdot \sigma^2 \cdot \dots \cdot \sigma^N$, and consider any $\sigma \geq \sigma^e$. If $n \leq N$, then obviously $V(\theta_n, \sigma) \leq e$.

If, however, $n > N$, consider the following identity:

$$(A) \left\{ \begin{aligned} &|\theta_n^n(\sigma') - \theta_n^n(s')| \leq |\theta_n^n(\sigma') - \theta^n(\sigma')| + |\theta^n(s') - \theta_n^n(s')| + \\ &+ |\theta^n(\sigma') - \theta_N^n(\sigma')| + |\theta_N^n(s') - \theta^n(s')| + |\theta_N^n(\sigma') - \theta_N^n(s')| \end{aligned} \right.$$

which holds for arbitrary choices of the $\theta_n^n(\sigma')$, $\theta^n(s')$, etc. In particular, then, we suppose that

- a) $\sigma', s' \geq \sigma$;
- b) σ' and s' are such that

$$|\theta_n^n(\sigma') - \theta_n^n(s')| > |V(\theta_n, \sigma) - e| \quad [\text{see 1.101}];$$

c) $\theta^n(\sigma')$ and $\theta^n(s')$ are chosen in accordance with b), 3.04, and then $\theta_N^n(\sigma')$ and $\theta_N^n(s')$ in accordance with a). We thus obtain from A) (under the hypothesis $n > N$, $\sigma \geq \sigma^e$)

$$V(\theta_n, \sigma) - e < |\theta_n^n(\sigma') - \theta_n^n(s')| \leq e + e + e + e + V(\theta_N, \sigma) \leq 5e,$$

and hence

$$6e \geq V(\theta_n, \sigma) \geq 0,$$

for $n > N$ and $\sigma \geq \sigma_0$. Since the same inequality obtains for $1 \leq n \leq N$, and since ϵ is arbitrary, we have the conclusion of the theorem.

3.06 Combining the results embodied in the theorems of 3.03 and 3.05, we may now state the following rather general

Theorem: If, on (c,d) , the sequence of set-functions $\{\theta_n\}$ converges to a set-function θ , and the limits $\lim_{\sigma} \theta_n(c,d)$ all exist, then

$$\lim_{\sigma} \theta(c,d) = \lim_n \lim_{\sigma} \theta_n(c,d),$$

provided only that either

a) the convergence is uniform,

or

b) the existence of the limits is uniform.

We use here the phrase "the limits exist uniformly" instead of saying that the corresponding sequence of σ -variation-functions approach θ uniformly, in the sense explained in 3.01.

CHAPTER II

A. THE STIELTJES INTEGRALS

4.01 Let f and α be two finite single-valued number-valued functions of x , where x is any point of the closed linear interval (a,b) . We consider any finite set of (at least two) points on (a,b) , which we denote collectively by σ :

$$\sigma \equiv \{x_0 < x_1 < \dots < x_m\}$$

and form either of the expressions

$$f\Delta\alpha \equiv (f, \alpha; \sigma) = \sum_1^m f(\xi_1) [\alpha(x_1) - \alpha(x_{1-1})]; \quad x_{1-1} \leq \xi_1 \leq x_1;$$

or

$$\bar{f}\Delta\alpha \equiv (m:f, \alpha; \sigma) = \sum_1^m f(\xi_1) [\alpha(x_1) - \alpha(x_{i-1})]; \quad x_{i-1} < \xi_1 < x_1.$$

The first of these expressions will be called a "Stieltjes sum" of f with respect to α on σ , and the second - a "modified Stieltjes sum" of f with respect to α on σ .

4.02. The Stieltjes sums have the following properties (most of which follow directly from the definitions):

S1 $(f, \alpha; \sigma)$ is a multiple-valued number-valued function on S , the set of all σ 's in (a,b) .

S2 If $f_3 = f_1 + f_2$, then for every value of $(f_3, \alpha; \sigma)$ there can be found a value of $(f_1, \alpha; \sigma)$ and $(f_2, \alpha; \sigma)$ such that

$$(f_3, \alpha; \sigma) = (f_1, \alpha; \sigma) + (f_2, \alpha; \sigma).$$

S3 If $\alpha_3 = \alpha_1 + \alpha_2$, then for every value of $(f, \alpha_3; \sigma)$ there can be found values of $(f, \alpha_1; \sigma)$ and $(f, \alpha_2; \sigma)$ - and conversely, to every pair of values, one for $(f, \alpha_1; \sigma)$ and one for $(f, \alpha_2; \sigma)$, there can be found a value of $(f, \alpha_3; \sigma)$ - such that

$$(f, \alpha_3; \sigma) = (f, \alpha_1; \sigma) + (f, \alpha_2; \sigma).$$

S4 If c is any constant, then to every value of $(f, \alpha; \sigma)$ there is a value of $(cf, \alpha; \sigma)$ [or a value of $(f, c\alpha; \sigma)$] - and conversely - such that

$$(cf, \alpha; \sigma) \text{ [or } (f, c\alpha; \sigma)] = c(f, \alpha; \sigma).$$

S5 If $\sigma^3 = \sigma^1 + \sigma^2$, in the sense explained in 1.03, then to every value of $(f, \alpha; \sigma^3)$ there corresponds a value of $(f, \alpha; \sigma^1)$ and a value of $(f, \alpha; \sigma^2)$ - and conversely - such that

$$(f, \alpha; \sigma^3) = (f, \alpha; \sigma^1) + (f, \alpha; \sigma^2).$$

Sol³ To every value of $(f, \alpha; \sigma)$ formed with a set σ whose initial element is c and whose final element is d ($a \leq c \leq d \leq b$) there exists a set σ' with the same end-elements and a value of $(\alpha, f; \sigma')$ formed with the set σ' such that

$$(f, \alpha; \sigma) + (\alpha, f; \sigma') = f(d)\alpha(d) - f(c)\alpha(c) \equiv k(c, d).$$

It should be noted here that if $c < e < d$, then clearly

$$k(c, e) + k(e, d) = k(c, d).$$

4.03 The modified Stieltjes sums obviously enjoy the properties S1 to S5 of the Stieltjes sums enumerated above. As regards a property analogous to S0, we leave the question open.

4.04 We conclude readily from the above considerations that if we specify f and α on (a, b) , we will have specified by the same token, in the expressions $f\Delta\alpha$ and $\bar{f}\Delta\alpha$, examples of a function $\theta(\sigma)$ of the sort contemplated in 1.04. The space S in this case will be the set of all those finite subsets of the continuum (a, b) which contain at least two elements.

Hence we lay down the following definitions:

Definition 1: The σ -limit of $(f, \alpha; \sigma)$ on (c, d) ($a \leq c \leq d \leq b$), when it exists, is called the σ -Stieltjes integral of f with respect to α on (c, d) , and is denoted by the symbol

$$\int_c^d f d\alpha.$$

Definition 2:¹⁴ The σ -limit of $(m; f, \alpha; \sigma)$ on (c, d) ($a \leq c \leq d \leq b$), when it exists, is called the modified σ -Stieltjes integral of f with respect to α on (c, d) and is denoted by the symbol

$$\bar{\int}_c^d f d\alpha.$$

In discussing the integrals defined here we shall refer to the sets σ as subdivisions of the interval determined by their end-elements, while the individual elements of a σ will be called division points of this interval. For example,

$\sigma \geq \sigma'$ will mean that σ and σ' are subdivisions of the same interval, and that every division point of this interval as given by σ' is also a division point as given by σ . The points ξ_1 which are used in the definitions of $f\Delta\alpha$ and $\bar{f}\Delta\alpha$ will be referred to as "inner" points.

4.05 The following¹⁵ facts about the integrals defined in 4.04 are readily deducible from the theory developed in Chap. I. Although they are stated for the σ -Stieltjes integral, they hold with obvious modifications, for the modified σ -Stieltjes integral as well.

$$\underline{J1} \quad \int_c^d d\alpha \equiv \int_c^d 1 \cdot d\alpha = \alpha(d) - \alpha(c).$$

J2 A necessary and sufficient condition for the existence of $\int_c^d f d\alpha$ is that for every $\epsilon > 0$ we can find a σ_ϵ , with end-elements c and d , so that

$$|(f, \alpha; \sigma') - (f, \alpha; \sigma'')| \leq \epsilon,$$

for all σ', σ'' finer than σ_ϵ .

J3 A necessary and sufficient condition for the existence of $\int_c^d f d\alpha$ is that for every $\epsilon > 0$ we can find a σ_ϵ , with end-elements c and d , so that

$$\underline{V(f, \alpha; \sigma) \leq \epsilon}$$

for all $\sigma \geq \sigma_\epsilon$; where $V(f, \alpha; \sigma)$ denotes the σ -variation of $(f, \alpha; \sigma)$ on σ , as explained in 1.08.

J4 If $\int_c^d f d\alpha$ exists, then for any σ on (c, d)

$$|\int_c^d f d\alpha - (f, \alpha; \sigma)| \leq V(f, \alpha; \sigma).$$

J5 If $\int_c^d f d\alpha$ exists, where f is bounded and α is of bounded total variation [$|f(x)| \leq M$ for all x on (c, d) and V is the total variation of α on (c, d)], then

$$|\int_c^d f d\alpha| \leq MV$$

Proof: We have

$$\begin{aligned} |(f, \alpha; \sigma)| &\leq \left| \sum_{\sigma} f(\xi_1) [\alpha(x_1) - \alpha(x_{1-1})] \right| \leq \\ &\leq \sum_{\sigma} |f(\xi_1)| |\alpha(x_1) - \alpha(x_{1-1})| \leq M \sum_{\sigma} |\alpha(x_1) - \alpha(x_{1-1})| \leq MV. \end{aligned}$$

It is clear now that, if the Stieltjes sums are all in absolute value (and for every determination) at most equal to a fixed number, then the absolute value of the σ -limit, when it exists, will not exceed the same number.

$$\begin{aligned} \underline{J6} \quad \int_c^d (f_1 + f_2) d\alpha &= \int_c^d f_1 d\alpha + \int_c^d f_2 d\alpha, \\ \int_c^d f d(\alpha_1 + \alpha_2) &= \int_c^d f d\alpha_1 + \int_c^d f d\alpha_2; \end{aligned}$$

provided, in each case, that the integrals on either side exist.

$$\underline{J7} \quad \int_c^d (kf) d\alpha = \int_c^d f d(k\alpha) = k \int_c^d f d\alpha,$$

provided any one of these integrals exist; where k is any constant $\neq 0$. If $k = 0$, we have to assume the existence of the rightmost integral.

J8 If $c < e < d$, then

$$\int_c^e f d\alpha + \int_e^d f d\alpha = \int_c^d f d\alpha$$

whenever the integrals on either side exist.

J9 If we define:

$$\int_c^d f d\alpha = - \int_d^c f d\alpha, \quad c \leq d; \quad \int_c^c f d\alpha = 0;$$

then, for any c, d, e [on (a, b)]

$$\int_c^d f d\alpha + \int_d^e f d\alpha + \int_e^c f d\alpha = 0,$$

whenever any two of the integrals exist.

[Neither J8 nor J9 holds for the "norm" - Stieltjes integrals.]

J10 If $\int_c^d f d\alpha$ exists, then $\int_c^{\bar{d}} f d\alpha$ also exists, and the two are equal.

Indeed, as explained in 2.12,

$$(m; f, \alpha; \sigma) < (f, \alpha; \sigma),$$

so that our theorem follows from 2.13.

4.06 The following property of the σ -Stieltjes integral does not hold for the modified integral:

$$\underline{J0} \quad \int_c^d f d\alpha + \int_c^d \alpha df = f(d)\alpha(d) - f(c)\alpha(c),$$

whenever either one of the integrals exists.

Proof: Because of property S0 of the Stieltjes sums, the set-functions $\theta_1(\sigma) \equiv (f, \alpha; \sigma)$ and $\theta_2(\sigma) \equiv (\alpha, f; \sigma)$ are related by parts, with $k(\gamma, \delta) = f(\cdot)\alpha(\delta) - f(\gamma)\alpha(\gamma)$. Hence, this theorem follows from 2.09.

Consider now, for the interval (c, d) , the case where

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

and

$$\alpha(x) = \begin{cases} 1 & \text{at the end-points} \\ 0 & \text{in the interior} \end{cases} \text{ of the interval.}$$

Here $(m:f;\alpha;\sigma)$ assumes the values $-1, 0, 1$ for any subdivision of (c, d) which contains at least three division points, while $(m:\alpha, f;\sigma) \equiv 0$. Hence $\int_c^d \alpha df = 0$, while the other integral does not even exist.

B. A SUFFICIENT CONDITION FOR THE EXISTENCE OF THE MODIFIED σ -STIELTJES INTEGRAL

5.01 By the "inner" oscillation of a function f on an interval (c, d) we shall mean the LUB of the set of numbers $|f(x_1) - f(x_2)|$, where x_1 and x_2 are any two points in the interior of (c, d) ; we shall denote this quantity by $\bar{O}(f, c, d)$. We will have occasion to use presently the following

Theorem: If $f(x)$ has discontinuities of the first kind only on the closed interval (a, b) , then for every $\epsilon > 0$ there exists a subdivision σ_ϵ , with end-elements a and b , such that the inner oscillation of f on any mesh of σ_ϵ is $\leq \epsilon$.

Proof: Since $f(x)$ has discontinuities of the first kind only, we can find,¹⁶ for any $\epsilon > 0$, a finite set of points

$$a' = y_0 < y_1 < \dots < y_n = b$$

such that at every other point on (a, b) the oscillation of $f(x)$ is $< \frac{\epsilon}{3}$. Considering now the interval (y_{i-1}, y_i) [$i = 1, \dots, n$], we redefine $f(x)$ at the two end-points thus:

$$f'(y_{i-1}) = f(y_{i-1} + 0), \quad f'(y_i) = f(y_i - 0)$$

(so that the redefined $f(x)$ may be double-valued at some points). Then $f(x)$ thus redefined, has at every point of the closed interval $I_i = (y_{i-1}, y_i)$ an oscillation which is $< \frac{\epsilon}{3}$. By including every point of I_i in an interval over which the oscillation of the redefined $f(x)$ does not exceed ϵ and applying the Borel theorem, we can obtain a division of I_i into a finite number of non-overlapping subintervals, over each of which the oscillation of the redefined $f(x)$, and so certainly the "inner" oscillation, does not exceed ϵ . Combining all these sub-intervals for $i = 1, \dots, n$ and noticing that the "inner" oscillation of $f(x)$ over each of these

subintervals is equal to the "inner" oscillation of the redefined $f(x)$ over the same intervals, we obtain the required subdivision of (a,b) .

5.02 We may now state the following

Theorem: If $f(x)$ has discontinuities of the first kind only on (a,b) , and $\alpha(x)$ is of bounded total variation there, then $\int_a^b f d\alpha$ exists.

Proof: Choose any $\epsilon > 0$, and let $\sigma\epsilon \equiv \{a = x_0 < \dots < x_m = b\}$ be a set such that $\bar{O}(f, x_{i-1}, x_i) \leq \epsilon$, $i = 1, \dots, m$. Let $\sigma \geq \sigma\epsilon$, say $\sigma = \sum_{i=1}^m s_i$, where $s_i \equiv \{x_{i-1} = x_{i0} < \dots < x_{ik_i} = x_i\}$, $i = 1, \dots, m$. Let $\{\xi_j^i\}$ ($j = 1, \dots, k_i$) be a system of inner points of s_i , and $\{\xi_i\}$ ($i = 1, \dots, m$) - such a system for $\sigma\epsilon$. Noticing that

$$\sum_{i=1}^m f(\xi_i) [\alpha(x_i) - \alpha(x_{i-1})] = \sum_{i=1}^m \sum_{j=1}^{k_i} f(\xi_j^i) [\alpha(x_{ij}) - \alpha(x_{i,j-1})],$$

we can now write

$$\begin{aligned} |(m:f, \alpha; \sigma)\xi' - (m:f, \alpha; \sigma\epsilon)\xi| &\leq \sum_{i=1}^m \sum_{j=1}^{k_i} |f(\xi_j^i) - f(\xi_i)| |\alpha(x_{ij}) \\ &\quad - \alpha(x_{i,j-1})| \leq \epsilon V \end{aligned}$$

where V is the total variation of α on (a,b) . Since the systems ξ, ξ' were arbitrary, we have for any $\sigma, \sigma' \geq \sigma\epsilon$:

$$\begin{aligned} |(m:f, \alpha; \sigma) - (m:f, \alpha; \sigma')| &\leq \\ \leq |(m:f, \alpha; \sigma) - (m:f, \alpha; \sigma\epsilon)| &+ |(m:f, \alpha; \sigma') - (m:f, \alpha; \sigma\epsilon)| \leq 2\epsilon V. \end{aligned}$$

Since V is fixed and ϵ is arbitrary, we have, by 1.07, the existence of $\int_a^b f d\alpha$.

C. A NECESSARY CONDITION FOR THE EXISTENCE OF THE INTEGRALS.

6.01 Ordinarily when the Stieltjes integral of f with respect to α is discussed, it is assumed that f is bounded; it will be noted that here the only requirement on f is that it be finite. Nonetheless, it can be shown that when $\int_a^b f d\alpha$ exists, then f is, in a certain sense, bounded; this will be indicated in the discussion to follow. It may be at once remarked that this discussion applies, with obvious changes, to the modified σ -Stieltjes integral as well.

6.02 We begin by demonstrating two lemmas.

Lemma 1: If to each point x of a closed and bounded linear point set F there corresponds a (closed or non-closed) interval I_x , having x as interior- or end-point; and if,

when x_0 is a limit-point from the left of points of F , I_{x_0} certainly extends to the left of x_0 ; and analogously for a right-hand limit-point; - then there exists a finite number of the intervals I_x such that every point of F is either an interior - or an end-point of one of these intervals.

Proof: Omitting details, it is fairly clear that the situation described in the lemma is reducible to the situation of the Borel theorem if we extend, wherever necessary, the intervals I_x in such a way that the extended portions do not include any points of F . Then, in the finite set of intervals whose existence is given by the Borel theorem, we may remove the extensions.

Lemma 2:17 If $\int_a^b f d\alpha$ exists, and f is unbounded on the right at β , $a \leq \beta < b$, then there exists $\gamma, \beta < \gamma \leq b$, such that $\alpha(x_1) = \alpha(\beta)$ for $\beta \leq x_1 \leq \gamma$; similarly, if f is unbounded on the left at β , $a < \beta \leq b$.

Proof: Let $\epsilon > 0$ be chosen, and let $\sigma\epsilon$, a subdivision of (a, b) , be such that for all $\sigma \geq \sigma\epsilon$,

$$a) \quad \left| (f, \alpha; \sigma) - \int_a^b f d\alpha \right| \leq \epsilon.$$

Let $\sigma\epsilon \geq \sigma\epsilon$ contain β as a division-point, and let γ' be the next following division-point of $\sigma\epsilon$. Consider now the term

$$b) \quad f(\xi) [\alpha(\gamma') - \alpha(\beta)], \quad \gamma' \leq \xi \leq \beta$$

of the Stieltjes sum $(f, \alpha; \sigma\epsilon)$. From a) we conclude that the value (which varies with the choice of ξ) of b) is bounded; however, $f(\xi)$ is unbounded in the set of all possible ξ 's, and so we must have $\alpha(\gamma') - \alpha(\beta) = 0$. By choosing appropriate σ 's such that $\sigma \geq \sigma\epsilon$ we can prove $\alpha(x) - \alpha(\beta) = 0$ for any x in (β, γ) , where γ is the division-point of $\sigma\epsilon$ next following β . A similar proof applies to the second part of the lemma.

6.03 We may now state the following

Theorem: If $\int_a^b f d\alpha$ exists, then there exists a finite (or empty) set of disjoint closed intervals

$$(1): i_1, i_2, \dots, i_n,$$

all lying in the interval (a, b) , such that α is constant on each interval of this set, and f is bounded on the complement of $\sum_{j=1}^n i_j$ in the interval (a, b) .

Proof: If f is bounded, then the empty set may be taken for the set (1). In the other case, the set F of points y such that f is unbounded in every neighborhood of y is a closed set. By the second lemma of 6.02, we may associate with every point $y \in F$ an interval I_y , which contains y as

interior- or end-point, such that α is constant on I_y . We further agree to take the largest possible such interval in each case (so that this interval need not be closed on either side). We thus have a set of points F and a set of intervals ($I_y/y < F$) satisfying the conditions of lemma 1, 6.02; for if the sequence $\{y_n\}$ approaches y , say, from the left, then obviously f is unbounded on the left at y and thus I_y extends to the left of y . From the method of construction, it is clear that two different intervals of our set do not overlap. Since, therefore, no reduction of this set is possible in the sense of lemma 1, it follows that this set contains only a finite number of intervals:

$$(I): I_1, I_2, \dots, I_n.$$

Consider now any one of these intervals, say I_k , whose end-points we may denote by a_k and b_k , $a_k < b_k$. If α is constant on the closed interval (a_k, b_k) , we put $I_k = i_k$. If, however, $\alpha(a_k)$ is different from the constant value of α on I_k , then it is clear that f is bounded on the right at a_k , so that we can find $a'_k > a_k$ such that no point of F is contained in (a_k, a'_k) . We proceed similarly with b_k , if necessary; we then put, in such a case, $(a'_k, b'_k) = i_k$. In this fashion we obtain a finite number of disjoint closed intervals

$$(i): i_1, \dots, i_n$$

such that α is constant on each of these intervals, and such that every point of F is either an interior - or an end-point of one of these intervals.

Finally, f is bounded on $B \equiv (a, b) - \sum_{j=1}^n i_j$. For, this will be so if f is bounded on $A \equiv (a, b) - \sum_{j=1}^n i_j$. Suppose therefore that there is a sequence of distinct points

$$x_1, \dots, x_k, \dots$$

all belonging to A , such that

$$|f(x_k)| > k; k = 1, \dots$$

This set of points has at least one limit-point y , and clearly $y < F$. Therefore y belongs to I_j for some j , and indeed it must be an end-point of I_j : say that it is the left-hand end-point. Now the method of construction of I_j shows that f is bounded on the left at y ; on the other hand, since y is a limit-point from the left of the sequence $\{x_k\}$ and since $|f(x_k)| > k$, we must conclude that f is unbounded on the left at y .

The theorem is thus fully demonstrated.

6.04 Let a_k and b_k be, respectively, the left-hand and right-hand end-points of the interval i_k of the theorem in 6.03; $k = 1, \dots, n$. Putting $a = b_0$, $b = a_{n+1}$, we have by repeated applications of J8, 4.05,

$$1) \int_a^b f d\alpha = \sum_{k=0}^n \int_{b_k}^{a_{k+1}} f d\alpha + \sum_{k=1}^n \int_{a_k}^{b_k} f d\alpha.$$

But α is constant on (a_k, b_k) , so that

$$2) \int_{a_k}^{b_k} f d\alpha = 0; \quad k = 1, \dots, n.$$

Combining 1) and 2) with the theorem of 6.03, we obtain the following

Theorem: If $\int_a^b f d\alpha$ exists, then there exists a finite number of disjoint closed intervals, all lying on (a, b) :

$$(j): j_1, \dots, j_{n+1}$$

such that f is bounded on (j) , α is constant in each interval of the complementary set, and

$$\int_a^b f d\alpha = \sum_{k=1}^{n+1} \int_{j_k} f d\alpha,$$

(where $\int_{j_k} f d\alpha \equiv \int_{a_k}^{b_k} f d\alpha$, a_k and b_k being the end-points of j_k).

Indeed, we need only put

$$j_1 = (b_0, a_1), \dots, j_{n+1} = (b_n, a_{n+1})$$

and take into account only those intervals j for which $a_{k+1} \neq b_k$.

D. THE EXISTENTIAL THEORY OF THE "NORM" - AND σ -INTEGRALS.

7.01 Corresponding to the two σ -Stieltjes integrals considered in the preceding sections, one also defines the so-called "norm" - integrals, i.e., integrals obtained as limits of Stieltjes sums when the passage to the limit is according to the "norm" - the maximum length of the several meshes - of σ . Thus, when the inner point of the i th mesh of σ is unrestricted (as in $f\Delta\alpha$, 4.01), we obtain the Riemann-Stieltjes (RS) integral;¹⁸ and, when the inner point is restricted (as in $\bar{f}\Delta\alpha$), we obtain what may be called the modified ($\bar{R}S$) Riemann-Stieltjes integral. If we agree to refer

to the σ -Stieltjes integral symbolically by (σ) , and to the modified σ -Stieltjes integral - by $(\bar{\sigma})$, then for a given pair of single-valued functions f and α , number-valued and finite on (a,b) , we may conceivably have one of the following 16 situations arising:

	(RS)	($\bar{RS}$)	(σ)	($\bar{\sigma}$)
1	+	+	+	-
2	+	+	-	+
3	+	-	+	+
4	+	+	-	-
5	+	-	+	-
6	-	+	+	-
7	+	-	-	+
8	+	-	-	-
9	-	+	-	-
10	-	-	+	-
11	-	+	+	+
12	+	+	+	+
13	-	+	-	+
14	-	-	+	+
15	-	-	-	+
16	-	-	-	-

where in 1, for example, we mean to say that $(RS) \int_a^b f d\alpha$, $(\bar{RS}) \int_a^b f d\alpha$, and $\int_a^b f d\alpha$ exist, while $\int_a^b f d\alpha$ does not exist. However, the following facts are known:

a) If (RS) exists, then all the other integrals exist (and are equal to it in value).

b) If $(\bar{RS})$ exists, then $(\bar{\sigma})$ also exists (and has the same value).

c) J10, 4.05.

d₁) If $(\bar{RS})$ exists, then f and α cannot have simultaneous discontinuities, except possibly at the end-points.

d₂)¹⁹ If (σ) exists, then f and α cannot have any simultaneous discontinuities from the same side.

d₃)²⁰ If f and α have no simultaneous discontinuities, and (σ) exists, then (RS) also exists (and has the same value).

Hence, in the table above, 1,2,3,4,5,7,8 are ruled out by a); b) rules out 6 and 9, and c) rules out 10).

As regards 11, if $(\bar{RS})$ and (σ) both exist, then f and α cannot have any simultaneous discontinuities, by d₁) and d₂); whence (RS) must also exist, by d₃); so that 11 is also an impossible situation.

The following examples show that the remaining five situations are indeed possible. In each case we shall take $(0,1)$ as our interval of definition.

$$\underline{12} \quad \alpha(x) \equiv 0.$$

$$\underline{13} \quad f(x) = \alpha(x) = \begin{cases} 0, & \text{when } x = 0; \\ 1, & \text{when } x > 0. \end{cases}$$

$$\underline{14}^{21} \quad f(x) = \begin{cases} 0, & \text{when } x < \frac{1}{2}; \\ 1, & \text{when } x \geq \frac{1}{2}; \end{cases}$$

$$\alpha(x) = \begin{cases} 0, & \text{when } x \leq \frac{1}{2}; \\ 1, & \text{when } x > \frac{1}{2}. \end{cases}$$

$$\underline{15} \quad \alpha(x) = f(x), \text{ where } f(x) \text{ is as in } \underline{14}.$$

$$\underline{16}^{22} \quad f(x) = \alpha(x) = \begin{cases} 0, & \text{when } x \text{ is rational;} \\ 1, & \text{when } x \text{ is irrational.} \end{cases}$$

E. ABOUT THE CHANGE OF THE INTEGRATING FUNCTION.

8.01 If the σ -Stieltjes integral of f with respect to α exists on (a, b) , then it follows from J8 and J9, 4.05, that $F(x) \equiv \int_a^x f d\alpha$ is a well-defined numerical function for all x on the interval (a, b) . This function $F(x)$ has the property that

$$F(x_1) - F(x_2) = \int_{x_1}^{x_2} f d\alpha.$$

8.02 Theorem:²³ Let the number-valued functions $f(x)$, $g(x)$, and $\alpha(x)$, x on (a, b) , be such that:

1) There is an $M > 0$ such that $|f(x)| \leq M$;

$a \leq x \leq b$;

2) $\int_a^b g d\alpha$ exists;

3) $\int_a^b f dF$ exists, where $F(x) \equiv \int_a^x g d\alpha$;

then:

a) $\int_a^y f g d\alpha$ exists, $a \leq y \leq b$;

b) $\int_a^y f g d\alpha = \int_a^y f dF$, $a \leq y \leq b$.

Proof: For all σ on (γ, δ) [$a \leq \gamma < \delta \leq b$], say

$$\delta \equiv \{\gamma = x_0 < \dots < x_r = \delta\},$$

we define the set-functions θ and $|\theta|$, of the sort described in 1.04, by putting for any corresponding set of inner points ξ ,

$$\theta\xi(\sigma) = (g, \alpha; \sigma)^\xi - \int_\gamma^\delta g d\alpha \equiv \sum_{i=1}^n \left\{ g(\xi_i) [\alpha(x_i) - \alpha(x_{i-1})] - \int_{x_{i-1}}^{x_i} g d\alpha \right\}$$

and

$$|\theta|^\xi(\sigma) = \sum_{i=1}^n \left| g(\xi_i) [\alpha(x_i) - \alpha(x_{i-1})] - \int_{x_{i-1}}^{x_i} g d\alpha \right|.$$

From the definition of the σ -Stieltjes integral, there is, for every $\epsilon > 0$, a σ^ϵ with end-elements γ and δ , such that for every $\sigma \geq \sigma^\epsilon$ we have

$$|(g, \alpha; \sigma) - \int_{\gamma}^{\delta} g d\alpha| \leq \epsilon;$$

hence, since $|\theta(\sigma)| = |(g, \alpha; \sigma) - \int_{\gamma}^{\delta} g d\alpha|$, we have

$$\lim_{\sigma} \theta(\gamma, \delta) = 0, \quad a \leq \gamma < \delta \leq b,$$

and so, by 2.11,

$$\lim_{\sigma} |\theta|(\gamma, \delta) = 0 \quad a \leq \gamma < \delta \leq b.$$

Otherwise expressed, we can find, for (a, y) in particular $[a \leq y \leq b]$, and for every $\epsilon > 0$, a σ^ϵ with end-elements a and y , such that

$$|\theta|(\sigma) \leq \frac{\epsilon}{M}; \quad \sigma \geq \sigma^\epsilon.$$

Consider now the set-function Ψ , defined for all σ on (a, y) by putting, for any corresponding set of inner points ξ ,

$$A) \quad \Psi^\xi(\sigma) = (f, F; \sigma)^\xi - (fg, \alpha; \sigma)^\xi.$$

We can write, for any $\sigma \geq \sigma^\epsilon$ and any ξ :

$$\begin{aligned} |\Psi^\xi(\sigma)| &= \left| \sum_{i=1}^n f(\xi_i) [F(x_i) - F(x_{i-1})] - \sum_{i=1}^n f(\xi_i) g(\xi_i) [\alpha(x_i) - \alpha(x_{i-1})] \right| \\ &= \left| \sum_{i=1}^n f(\xi_i) \left[\int_{x_{i-1}}^{x_i} g d\alpha - g(\xi_i) [\alpha(x_i) - \alpha(x_{i-1})] \right] \right| \leq \\ &\leq M \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} g d\alpha - g(\xi_i) [\alpha(x_i) - \alpha(x_{i-1})] \right| \leq M |\theta|^\xi(\sigma) \leq \epsilon, \end{aligned}$$

or

$$\lim_{\sigma} \Psi(a, y) = 0, \quad a \leq y \leq b.$$

By taking into account A), and observing that $\lim (f, F; a, y)$ exists, we obtain finally the theorem.

The above theorem need not hold if f is merely finite. Thus, let

$$g(x) = \begin{cases} 1 - 2x, & 0 \leq x \leq \frac{1}{2}, \\ 1, & \frac{1}{2} < x \leq 1; \end{cases}$$

$$\alpha(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq x \leq 1; \end{cases}$$

for $x \neq \frac{1}{2}$, $f(x) = \frac{1}{g(x)}$ [so that $f(x)$ is unbounded on $(0,1)$];
 $f(\frac{1}{2}) = 1$.

Here

$$F(x) = \int_0^x g d\alpha \equiv 0, \quad 0 \leq x \leq 1,$$

so that

$$\int_0^y f dF \equiv 0, \quad 0 \leq y \leq 1;$$

however

$$\int_0^y f g d\alpha \text{ does not exist.}$$

8.03 It is fairly clear that the theorem in 8.02 holds for the modified σ -Stieltjes integral as well.

F. ANOTHER EXAMPLE OF A SET-FUNCTION

9.01 Let $K(x,y)$ be a single-valued number-valued finite function of the two variables x and y , where $a \leq x \leq b$, $a \leq y \leq b$. Choosing any finite set σ of numbers $x_0, \dots, x_m$, where

$$\sigma \equiv \{a = x_0 < x_1 < \dots < x_m = b\},$$

consider expressions of the type

$$L(\sigma) \equiv \sum_{\sigma} K(x,x) = \sum_1^m [K(\xi_1, x_1) - K(\xi_1, x_{i-1})],$$

where ξ_1 , for every i , is an arbitrary number in the i th mesh of σ . Thus $L(\sigma)$ is a many-valued set-function, and it is fairly easy to see that it is an additive set-function.

We may therefore consider the set-function L in the light of Chap. I; in particular, if $\lim_{\sigma} L(a,b)$ exists (as a finite number), we define²⁴

$$\int_a^b d_1 K(x,x) \equiv \lim_{\sigma} L(a,b),$$

and call this σ -limit the Stieltjes integral of $K(x,y)$ on (a,b) .

9.02 We proceed to enumerate some of the properties of this integral. [The numbering is such as to correspond with the numbering of the analogous properties of the Stieltjes integral; see J0, J1 - J10 in 4.05 and 4.06.]

K1 If $K(x,y) \equiv g(x)$, then $\int_a^b d_1 K(x,x) = 0$.
 If $K(x,y) \equiv h(y)$, then $\int_a^b d_1 K(x,x) = h(b) - h(a)$.

K2 If $\int_a^b d_1 K(x,x)$ exists, then for every $\epsilon > 0$ one can find a σ^ϵ , with end-elements a and b , such that for all $\sigma', \sigma'' \geq \sigma^\epsilon$ we have

$$|L(\sigma') - L(\sigma'')| \leq \epsilon.$$

K3 If $\int_a^b d_1 K(x,x)$ exists, then for every $\epsilon > 0$ one can find a σ^ϵ , with end-elements a and b , such that for all $\sigma \geq \sigma^\epsilon$

$$V(L, \sigma) \leq \epsilon;$$

where $V(L, \sigma)$ denotes the σ -variation of L on σ (see 1.08).

K4 If $\int_a^b d_1 K(x,x)$ exists, then

$$|\int_a^b d_1 K(x,x) - L(\sigma)| \leq V(L, \sigma) \text{ for all } \sigma.$$

K6 If we put $K_3(x,y) = K_1(x,y) + K_2(x,y)$, and if $\int_a^b d_1 K_1(x,x)$ and $\int_a^b d_1 K_2(x,x)$ both exist, then $\int_a^b d_1 K_3(x,x)$ also exists, and

$$\int_a^b d_1 K_3(x,x) = \int_a^b d_1 K_1(x,x) + \int_a^b d_1 K_2(x,x).$$

K8 If $a < c < b$, then

$$\int_a^b d_1 K(x,x) = \int_a^c d_1 K(x,x) + \int_c^b d_1 K(x,x),$$

whenever the integrals on either side exist.

K11 If $K(x,y) \equiv g(x) h(y)$, then

$$\int_a^b d_1 K(x,x) = \int_a^b g dh,$$

whenever the integrals on either side exist.

K12 If the two functions $K(x,y)$ and $K_1(x,y)$ agree in their values for all points $P: (x,y)$ in some ϵ -neighborhood of the diagonal from a to b , then

$$\int_a^b d_1 K(x,x) = \int_a^b d_1 K_1(x,x),$$

whenever either one of the integrals exists.

An ϵ -neighborhood of the diagonal from a to b means here the set of all points $P: (x,y)$ of the fundamental square whose distance from the straight joining $P: (a,a)$ to $P: (b,b)$ is less than the positive number ϵ .

9.03 Having given a function $K(x,y)$ and the corresponding set-function L , we consider also the function $\mathcal{K}(x,y)$,

defined on the fundamental square by putting

$$\mathcal{K}(x, y) = K(y, x),$$

and its corresponding set-function $\bar{L}$.

If we now put (when the necessary σ -limit exists)

$$\int_a^b d_1 \mathcal{K}(x, x) = \int_a^b d_2 K(x, x),$$

then the following property is true:

$$\underline{KO} \quad \int_a^b d_1 K(x, x) + \int_a^b d_2 K(x, x) = K(b, b) - K(a, a),$$

whenever either one of the integrals exists.

Proof: Let

$$\sigma = \{x_0 < x_1 \dots < x_m\}; \quad a \leq x_0 < x_m \leq b;$$

and let

$$L^o(\sigma) = \sum_1^m [K(\xi_1, x_1) - K(\xi_1, x_{1-1})]$$

be any one value of L for this σ . Putting $\xi_0 = x_0$, $\xi_{m+1} = x_m$, we can write

$$\begin{aligned} \sum_{i=1}^m [K(\xi_i, x_i) - K(\xi_i, x_{i-1})] &= - \sum_{i=1}^{m+1} [K(\xi_i, x_{i-1}) - K(\xi_{i-1}, x_{i-1})] + \\ &\quad + K(x_m, x_m) - K(x_0, x_0) = \\ &= - \sum_{i=1}^{m+1} [\mathcal{K}(x_{i-1}, \xi_i) - \mathcal{K}(x_{i-1}, \xi_{i-1})] + K(x_m, x_m) - K(x_0, x_0). \end{aligned}$$

Now it is fairly clear that $\sum_{i=1}^{m+1} [\mathcal{K}(x_{i-1}, \xi_i) - \mathcal{K}(x_{i-1}, \xi_{i-1})]$ is one of the values of the set-function $\bar{L}$ for the set

$$\sigma' = \{\xi_0 < \xi_1 < \dots < \xi_{k+1}\}$$

[which may be obtained from the set

$$\xi_0 \leq \xi_1 \leq \dots \leq \xi_{m+1}$$

by discarding one of every two consecutive ξ 's which may be identical]; in other words, if we put

$$k(\alpha, \beta) = K(\beta, \beta) - K(\alpha, \alpha),$$

we see that the set-functions L and $\bar{L}$ are related by parts on (a, b) . KO follows therefore from the theorem of 2.09.

9.04 If, in 9.01, we restrict the "inner" points ξ to lie in the interior of the i th mesh of σ , then the sums $\sum_{\sigma} K(x, x)$ will give rise to what may be termed the "modified" Stieltjes integral of $K(x, y)$ on (a, b) . It is fairly easy to see that, except for K_0 , all the other properties of $\int_a^b d_1 K(x, x)$ continue to hold for this "modified" integral.

CHAPTER III

CONCERNING SEQUENCES OF INTEGRALS

10.01 Suppose there is given a function α and a sequence of functions $\{f_n\}$, all on (c,d) , and suppose that the variation-functions $V(f_n, \alpha, \sigma)$ approach 0 uniformly on (c,d) (see 3.01). Since the convergence of these variations to 0 uniformly implies in particular their σ -convergence to 0, which in its turn, by 1.10, implies the existence of $\int_c^d f_n d\alpha$ for all n , we will also use the expression "the integrals $\int_c^d f_n d\alpha$ exist uniformly" instead of $\lim_{\sigma} V(f_n, \alpha; c, d) = 0$ uniformly.

10.02 Theorem: On (c,d) , let

- 1) $f_n(x) \rightarrow f(x)$,
 2) the integrals $\int_c^d f_n d\alpha$ exist uniformly;

[where $\alpha(x)$ and $f(x)$ are finite]

then

- a) $\int_c^d f d\alpha$ exists,
 b) $\lim_n \int_c^d f_n d\alpha$ exists,
 c) $\lim_n \int_c^d f_n d\alpha = \int_c^d f d\alpha$.

Proof: Consider any value of $(f, \alpha; \sigma)$ for any given σ , say:

$$A \equiv (f, \alpha; \sigma)^0 = \sum_1^m f(\xi_i) [\alpha(x_i) - \alpha(x_{i-1})].$$

Take the sequence of numbers A_n , where

$$A_n = \sum_1^m f_n(\xi_i) [\alpha(x_i) - \alpha(x_{i-1})]$$

and put $P = \sum_1^m |\alpha(x_i) - \alpha(x_{i-1})|$, so that P , while it depends on m , is independent of n . Choose any $\epsilon > 0$; by 1), there exists N_ϵ , so that, for $n \geq N_\epsilon$ and for $i = 1, \dots, m$

$$|f(\xi_i) - f_n(\xi_i)| \leq \frac{\epsilon}{P}.$$

We may now write, for $n \geq N_\epsilon$:

$$\begin{aligned} |A_n - A| &\leq \sum_1^m |f(\xi_i) - f_n(\xi_i)| |\alpha(x_i) - \alpha(x_{i-1})| \leq \\ &\leq \frac{\epsilon}{P} \sum_1^m |\alpha(x_i) - \alpha(x_{i-1})| = \epsilon. \end{aligned}$$

Hence, the set-functions determined by the f_n and α converge, in the sense explained in 3.01, to the set-function determined by f and α ; our theorem is thus a direct consequence of 3.03.

10.03 A sequence of integrals of the sort defined in 9.01 or 9.04 has the same property as above; we state the theorem in terms of the first-named integrals:

Theorem: On the fundamental square, let

1) $\{K_n(x, y)\}$ be a sequence of functions which converge to a function $K(x, y)$ at every point $P: (x, y)$ of the fundamental square;

2) the integrals $\int_a^b d_1 K_n(x, x)$ exist uniformly (see 3.06).

Then

a) $\int_a^b d_1 K(x, x)$ exists;

b) $\lim \int_a^b d_1 K_n(x, x)$ exists;

c) $\int_a^b d_1 K(x, x) = \lim_n \int_a^b d_1 K_n(x, x)$.

Proof: Let $\epsilon > 0$ be chosen; if σ is any subdivision of (a, b) and $L^0(\sigma)$ - some value of L for it, say

$$L^0(\sigma) = \sum_1^m [K(\xi_1, x_1) - K(\xi_1, x_{1-1})],$$

then we can find a positive integer N_ϵ for which

$$n \geq N_\epsilon \left\{ \begin{array}{l} |K(\xi_1, x_1) - K_n(\xi_1, x_1)| \leq \frac{\epsilon}{2M} \\ |K(\xi_1, x_{1-1}) - K_n(\xi_1, x_{1-1})| \leq \frac{\epsilon}{2M} \end{array} \right\}; i = 1, \dots, m.$$

Hence, for $n \geq N_\epsilon$,

$$\begin{aligned} |L^0(\sigma) - L_n^0(\sigma)| &= \left| \sum_{i=1}^m [K(\xi_i, x_i) - K(\xi_i, x_{i-1})] - \sum_{i=1}^m [K_n(\xi_i, x_i) - K_n(\xi_i, x_{i-1})] \right| \\ &\leq \sum_{i=1}^m |K(\xi_i, x_i) - K_n(\xi_i, x_i)| + \sum_{i=1}^m |K(\xi_i, x_{i-1}) - K_n(\xi_i, x_{i-1})| \\ &\leq M \cdot \frac{\epsilon}{2M} \cdot 2 = \epsilon, \end{aligned}$$

which shows that the sequence of set-functions $\{L_n\}$ converges to the set-function L . Because of 2), our theorem is thus a consequence of 3.06.

Because of property K12 of the integrals treated here, it is obvious that hypothesis 1) of our theorem need hold only for some ϵ -neighborhood of the diagonal of the fundamental square.

10.04 Theorem: On (c, d) , let

1) $\alpha(x)$ be of bounded total variation,

- 2) $f_n(x) \rightarrow f(x)$ uniformly,
 3) $\int_c^d f_n d\alpha$ exist for all n ;

then

- a) $\int_c^d f d\alpha$ exists,
 b) $\lim_n \int_c^d f_n d\alpha = \int_c^d f d\alpha$.

Proof: Let V_γ^δ denote the total variation of α on (γ, δ) , where $c \leq \gamma < \delta \leq d$, and put $V_c^d = M$. Let $\epsilon > 0$ be chosen, and let N be such that, for all x on (c, d) ,

$$|f_n(x) - f(x)| \leq \frac{\epsilon}{M}; \quad n \geq N.$$

Consider now, for any $n \geq N$, and any σ with end-elements γ and δ , the expressions

$$\sum_1^n f(\xi_1) [\alpha(x_1) - \alpha(x_{i-1})] \quad \text{and} \quad \sum_1^n f_n(\xi_1) [\alpha(x_1) - \alpha(x_{i-1})].$$

We have

$$\begin{aligned} & \left| \sum_1^n f(\xi_1) [\alpha(x_1) - \alpha(x_{i-1})] - \sum_1^n f_n(\xi_1) [\alpha(x_1) - \alpha(x_{i-1})] \right| \leq \\ & \leq \sum_1^n |f(\xi_1) - f_n(\xi_1)| |\alpha(x_1) - \alpha(x_{i-1})| \leq \frac{\epsilon}{M} \cdot V_\gamma^\delta \leq \frac{\epsilon}{M} \cdot M = \epsilon. \end{aligned}$$

But this means that the sequence of set-functions $\{\theta_n\}$, as determined by α and the sequence $\{f_n\}$, is uniformly convergent to the set-function θ , determined by α and f . Thus the hypotheses of 3.06 are all satisfied here, and our theorem is proved.

10.05 A further modification of the theorem 10.02 is contained in the following

Theorem:²⁵ On (c, d) , let there be given (finite) functions $\alpha(x)$ and $f(x)$ and a sequence of functions $\{f_n(x)\}$ such that

- a) in every subinterval of (c, d) there is a sequence of points $\{x_n\}$ converging to some point x such that $f_n(x_n) \rightarrow f(x)$;
 b) the integrals $\int_c^d f_n d\alpha$ exist uniformly;
 c) $\int_c^d f d\alpha$ exists;

then

1) $\lim_n \int_c^d f_n d\alpha$ exists

and

2) $\int_c^d f d\alpha = \lim_n \int_c^d f_n d\alpha$.

Proof: We consider the set function $\bar{\theta}$, whose set of values for a given σ on (c, d) is given by the set of numbers $\sum_0 f(\xi_1) \Delta_1 \alpha$ where, for any i , ξ_1 is restricted to be a point

for which there is, in (x_{i-1}, x_i) , a sequence $\{\xi_{in}\}$ in accordance with a). It is clear that $\bar{\theta}$ is an additive set-function. By an argument similar to the one in the proof of 10.02, we can see that the set-functions θ_n , given by the f_n and α , are such that $\theta_n \rightarrow \bar{\theta}$, and hence, by the theorem of 10.02.

$$\lim_n \int_c^d f_n d\alpha = \lim_{\sigma} \bar{\theta}(c, d).$$

But obviously $\bar{\theta} < \theta$, where θ is the set-function given by f and α , and so, by 2.13

$$\lim_{\sigma} \bar{\theta}(c, d) = \lim_{\sigma} \theta(c, d).$$

Hence the theorem.

10.06 Theorem: 26 Let $\{a_n(x)\}$ be a sequence of functions, uniformly of bounded total variation on (a, b) , which converges to a function $\alpha(x)$, also of bounded total variation on (a, b) , at a set of points E , which includes a and b and is dense on (a, b) ; let $f(x)$ be continuous on (a, b) . Then

$$\lim_n \int_a^b f d\alpha_n = \int_a^b f d\alpha.$$

Proof: Because of their existence as Riemann-Stieltjes integrals, the integrals in question certainly exist and so, by 4.06, also the integrals $\int_a^b \alpha df$ and $\int_a^b \alpha_n df$, for all n . Hence, if the last named integrals exist uniformly, all the conditions of the theorem in 10.05 are satisfied, and we will have

$$\lim_n \int_a^b \alpha_n df = \int_a^b \alpha df.$$

From this we obtain, using 4.06 again:

$$\begin{aligned} \lim_n [\alpha_n(b)f(b) - \alpha_n(a)f(a) - \int_a^b f d\alpha_n] &= \alpha(b)f(b) \\ &- \alpha(a)f(a) - \int_a^b f d\alpha, \end{aligned}$$

and since

$$\lim_n \alpha_n(b) = \alpha(b), \quad \lim_n \alpha_n(a) = \alpha(a),$$

we finally obtain the theorem.

To prove that the $\int_a^b \alpha_n df$ exist uniformly, we proceed as follows:

Since $f(x)$ is continuous on (a, b) , then for every $\epsilon > 0$ we can find a positive number δ such that, if $|x_1 - x_2| \leq \delta$,

then

$$|f(x_1) - f(x_2)| \leq \frac{e}{2M},$$

where M is a positive number exceeding, for all n , the total variation of $\alpha_n(x)$ on (a, b) .

Let $\sigma^0 \equiv \{a = y_0 < \dots < y_m = b\}$ be a subdivision of (a, b) for which the length of the longest mesh is $\leq \delta$; let σ^1 and s^1 be two subdivisions of the i th mesh of σ^0 , with corresponding sets of inner points ξ and ξ' ; finally, let x_1 be any point in (y_{i-1}, y_i) , and put $f_1 = f(x_1)$. We may now write:

$$\begin{aligned} (f, \alpha_n; \sigma^1) \xi - (f, \alpha_n; s^1) \xi' &= (f, \alpha_n; \sigma^1) \xi - (f_1, \alpha_n; \sigma^1) + \\ &+ (f_1, \alpha_n; s^1) - (f, \alpha_n; s^1) \xi' = (f - f_1, \alpha_n; \sigma^1) \xi - (f - f_1, \alpha_n; s^1) \xi'. \end{aligned}$$

Each of the Stieltjes sums on the right is composed of terms of the form

$$[f(\xi_{1j}) - f(x_1)] \Delta_{1j} \alpha$$

which in absolute value are $\leq \frac{e}{2M} \cdot |\Delta_{1j} \alpha|$, since $|\xi_{1j} - x_1| \leq \delta$.

$$\begin{aligned} \text{Hence } |(f, \alpha_n; \sigma^1) \xi - (f, \alpha_n; s^1) \xi'| &\leq \frac{e}{2M} \left\{ \sum_{\sigma^1} |\Delta_{1j} \alpha| + \sum_{s^1} |\Delta'_{1j} \alpha| \right\} \leq \\ &\leq 2 \cdot \frac{e}{2M} \cdot V_{1n}, \end{aligned}$$

where V_{1n} is the total variation of α_n on the i th mesh of σ^0 . Summing this with respect to i , we get

$$\begin{aligned} V(f, \alpha_n; \sigma^0) &= \sum_{i=1}^m V(f, \alpha_n; y_{i-1}, y_i) = \\ &= \sum_{i=1}^m \text{LUB}_{\sigma^1, s^1, \xi, \xi'} |(f, \alpha_n; \sigma^1) \xi - (f, \alpha_n; s^1) \xi'| \leq \frac{e}{M} \cdot \sum_{i=1}^m V_{1n} \leq \\ &\leq \frac{e}{M} \cdot M = e. \end{aligned}$$

Since this is true for all n as soon as $\sigma^0 \geq \sigma^e$, where σ^e is a subdivision of (a, b) the length of whose longest mesh is $\leq \delta$, and since

$$V(f, \alpha_n; \sigma) = V(\alpha_n, f; \sigma),$$

by 2.08, we finally have

$$\lim_{\sigma} V(\alpha_n, f; a, b) = 0 \text{ uniformly.}$$

10.07 In the discussions of 10.02, 10.04 and 10.05 we have used only such properties of the σ -Stieltjes integrals which hold for the modified σ -Stieltjes integrals as well; hence, the theorems in those paragraphs hold for these integrals also. Because of J10, 4.05, the theorem in 10.06 likewise holds for the modified σ -Stieltjes integrals.

FOOTNOTES

The following abbreviations will be used:

I. S. Pollard, The Stieltjes Integral and its Generalizations, Quarterly Journal of Mathematics, v 49(1920-3), pp. 73-138.

II. H. L. Smith, On the Existence of the Stieltjes Integral, Transactions of the American Mathematical Society, v 27(1925), pp. 491-515.

III. H. E. Bray, Elementary Properties of the Stieltjes Integral, Annals of Mathematics, v 20, pp. 177-186.

- 1) See The Algebra of Many-Valued Quantities, Mathematische Annalen, v. 104(1931) pp. 260-290; Many-Valued Riemann-Stieltjes Integration, I and II, Cambridge Philosophical Society Proceedings, v 27(1931) pp. 325-380.
- 2) See II.
- 3) I.
- 4) See Untersuchungen Ueber den Integralbegriff, Mathematische Annalen, v 103 (1930), pp. 654-696.
- 5) This definition was first given by Pollard; see II, p. 90.
- 6) See The Fredholm Theory of Stieltjes Integral Equations, Annals of Mathematics, v. 25, pp. 142-158.
- 7) Compare here the first of Miss Young's papers cited above.
- 8) See Moore & Smith, A General Theory of Limits, American Journal of Mathematics, v 44(1922), pp. 102-121.
- 9) Loc. cit., sec. 2.
- 10) Read: least upper bound.
- 11) See II, sec. 2.
- 12) Cf. Whittaker, On the Stieltjes Integral, Proceedings of the Edinburgh Mathematical Society, Ser. 2, vl(1929), p. 212.
- 13) See III, p. 185.
- 14) Cf. I, sec. 12.
- 15) Cf. I, sec. 6.
- 16) See Hobson, Functions of a Real Variable, 2nd Ed., vl, p. 286.
- 17) Cf. II, sec. 3.
- 18) As defined, for example, in III, p. 177.
- 19) See II, sec. 3.
- 20) L.C., sec. 4.
- 21) Cf. I, p. 80.
- 22) Cf. II, sec. 4.

- 23) See Whittaker's paper cited in 12), p. 214.
- 24) See Fischer's paper cited in 6).
- 25) See Hildebrandt, Convergence of Sequences of Linear Operations, Bulletin of the American Mathematical Society, v 28(1922), pp. 53-58.
- 26) L.C., p. 57.

